Plan:

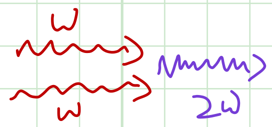
(1) Finish non-linear optics

(2) Diagrammatic Perturbation theory + diagrams

Last time:

$P = \epsilon_0 \chi E + \chi^{(2)} E^2 + \dots$

$\Rightarrow \chi^{(2)}$ term can be used for freq. doubling.

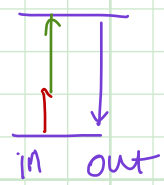


$\Rightarrow$ Generally $\chi^{(2)}$ non-linearity can "mix" 3 frequencies

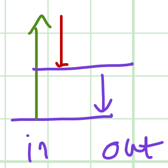
$\omega_1 \pm \omega_2 \pm \omega_3 = 0 \Rightarrow$ "3-wave mixing"

$\Rightarrow$ this is also comonly used in optics

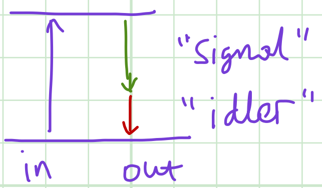
Sum freq generator



diff. freq. generator



Parametric down conv.

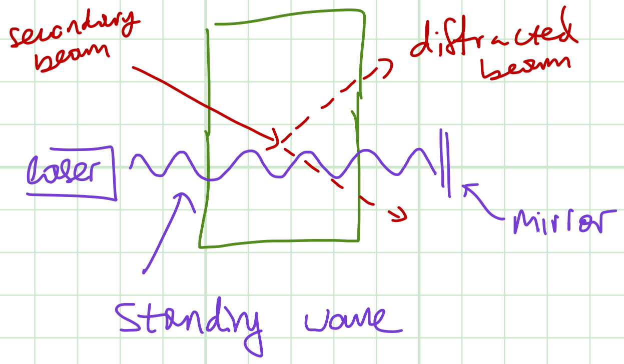


Alternative way to think about non-linearity (4-beams)

$n(E) = \sqrt{1 + \chi + 2\frac{\chi^{(2)}}{\epsilon_0} E + \dots}$

This is like a difraction grating

with $\Delta n \sim E$

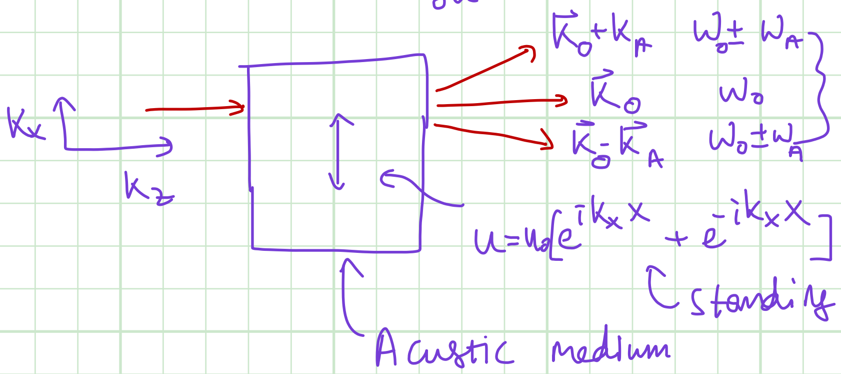


Acusto-optics + the AOM [Acusto-optical modulator]

Instead of using light to make a diffraction grating [or to give a kick] we can use ultrasound

$\Rightarrow$ χ depend on strain u, which comes from the sound waves

$$\chi = \chi + \frac{\partial \chi}{\partial u} u$$



must conserve freq + momentum

other outputs?

The above device is called an AOM or Bragg cell

$\Rightarrow$ used in telecom + in optics / cold atoms experiments

$\Rightarrow$ fast control of freq. shift + output intensity

$\Rightarrow$ mode locking, imaging systems for UCA, lattice systems for UCA

intensity + phase shift.

$\Rightarrow$ conveyor belt

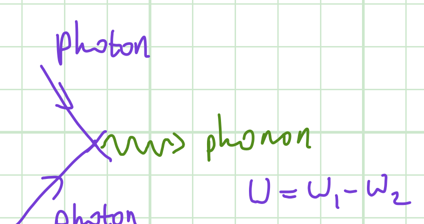
Raman scattering: AOM in reverse

$\Rightarrow$ Hard to directly couple photons to phonons due to large mismatch in freq.

$\Rightarrow$ use a 2-photon process

$$H' = P \cdot E = \text{linear part} + \epsilon_0 u E^2 \frac{\partial \chi}{\partial u}$$

$$a_K \, a_q \, c^{\dagger}_{K+q}$$



$\omega = \omega_1 - \omega_2$

$K = \vec{K}_1 + \vec{K}_2$

Diagrammatic Perturbation Theory

(1) Interaction picture

Assume that

$H = H_0 + V$

where

- H_0 part we can "solve" exactly
- $[H_0, V] \neq 0$, otherwise problem is "trivial"

Operators $\hat{O}(t) = e^{iH_0 t} O_{\text{Schrodinger}} e^{-iH_0 t}$

Wave functions $\hat{\Psi}(t) = e^{iH_0 t} e^{-iHt} \Psi(0)$ ← note: $\hat{\Psi}(t) \neq e^{i(H_0 - H)t} = e^{iVt}$

since $[H_0, V] \neq 0$

Quick check that we can compute matrix elements + they are what we expect:

$$\langle \hat{\Psi}_1^\dagger(t)\, \hat{O}(t)\, \hat{\Psi}_2(t) \rangle = \underbrace{\langle \Psi_1(0)\, e^{iHt} e^{-iH_0 t}}_{\hat{\Psi}_1^\dagger(t)} \underbrace{e^{iH_0 t} O_{\text{Schrodinger}} e^{-iH_0 t}}_{\hat{O}(t)} \underbrace{e^{iH_0 t} e^{-iHt} \Psi_2(0) \rangle}_{\hat{\Psi}_2(t)}$$

$$= \langle \Psi_1(0)\, e^{iHt} O_{\text{Schrodinger}} \underbrace{e^{-iHt} \Psi_2(0)}_{\Downarrow} \rangle = \langle \Psi_1(t)\, O_{\text{Schrodinger}}\, \Psi_2(t) \rangle$$

$\Psi_2(t)$ in the Schrödinger picture

(2) Time dependence

Time dependence of operators is governed by $H_0 \Rightarrow$ easy

Time dependence of wave functions is more complicated ...

$$\partial_t \hat{\Psi}(t) = \partial_t\, e^{iH_0 t} e^{-iHt} \Psi(0) = \left(i H_0 e^{iH_0 t} e^{-iHt} - i e^{iH_0 t} H e^{-iHt} \right) \Psi(0)$$

H_0 commutes with itself

$$= i e^{iH_0 t} [H_0 - H] e^{-iHt} \Psi(0)$$

$$= -i e^{iH_0 t} V e^{-iHt} \Psi(0)$$

$$= -i e^{iH_0 t} V e^{-iH_0 t} [e^{iH_0 t} e^{-iHt} \Psi(0)] = -i \hat{V}(t) \hat{\Psi}(t)$$

$$\partial_t \hat{\Psi}(t) = -i \hat{V}(t) \hat{\Psi}(t)$$

(3) Dyson eq:

Let us introduce the unitary time evolution operator

$U(t) = e^{iH_0 t} e^{-iHt}$

This operator is important because it takes $\hat{\Psi}(0) \rightarrow \hat{\Psi}(t) = U(t) \hat{\Psi}(0)$

Following above arguments

$$\partial_t U(t) = i e^{iH_0 t}(H_0 - H) e^{-iHt} = -i e^{iH_0 t} V e^{-iH_0 t} e^{iH_0 t} e^{-iHt} = -i \hat{V}(t) \; U(t)$$

Formally integrating this equation we obtain:

$$\int_0^t [\partial_{t'} U(t')] dt' = \int_0^t -i \hat{V}(t') U(t') \, dt'$$

$$= U(t) - U(0) = -i \int_0^t \hat{V}(t') U(t') \, dt'$$

this is identity

Hence we obtain:

$$U(t) = 1 - i \int_0^t \hat{V}(t') \; U(t') \; dt'$$

which we can solve by iteration

$$U(t) = 1 - i \int_0^t \hat{V}(t_1) dt_1 + (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \; \hat{V}(t_1) \hat{V}(t_2) + \ldots$$

$$= \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \; \hat{V}(t_1) \hat{V}(t_2) \ldots \hat{V}(t_n) \quad \equiv \text{Dyson Series}$$

(4) Time ordering: This expression is inconvenient due to integrals having different upper bounds. Let us "fix" it by introducing the time ordering operator T [not to be confused with temperature]

$$T[\hat{V}(t_1) \hat{V}(t_2) \hat{V}(t_3)] = [\text{puts times in order from largest to smallest}]$$

$$= \hat{V}(t_3) \; \hat{V}(t_2) \; \hat{V}(t_1) \quad \text{if} \quad t_3 > t_2 > t_1$$

Using the time ordering operator we can rewrite the Dyson series as

note all upper limits are same

$$U(t) = 1 - i \int_0^t dt_1 \, V(t_1) + \frac{(-i)^2}{2!} \int_0^t dt_1 \int_0^t dt_2 \; T[V(t_1) V(t_2)] + \ldots$$

these fix answer

How did that work? Let us go through the mechanics of the exercise slowly

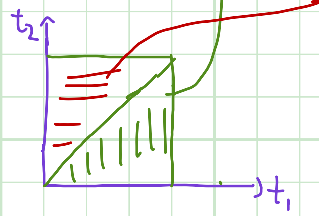
$$T[\hat{V}(t_1) \hat{V}(t_2)] = \Theta(t_1 - t_2) \hat{V}(t_1) \hat{V}(t_2) + \Theta(t_2 - t_1) \hat{V}(t_2) \hat{V}(t_1)$$

commute operators to time order them

$$\frac{(-i)^2}{2!}\int_0^t dt_1 \int_0^t dt_2 \left[\Theta(t_1-t_2)\hat{V}(t_1)\hat{V}(t_2) + \Theta(t_2-t_1)\hat{V}(t_2)\hat{V}(t_1)\right]$$

$$= \frac{(-i)^2}{2!}\left[\int_0^t dt_1 \int_0^{t_1} dt_2 \; \hat{V}(t_1)\hat{V}(t_2) + \int_0^t dt_1 \int_{t_1}^t dt_2 \; \hat{V}(t_2)\hat{V}(t_1)\right] = \frac{(-i)^2}{2!}\, 2\int_0^t dt_1 \int_0^{t_1} dt_2 \; \hat{V}(t_1)\hat{V}(t_2)$$

(QED)



these are dummy variables ⇒ swap $t_1 \leftrightarrow t_2$

⇒ integrals become same

The time ordering procedure works for all terms of the Dyson series

Hence we can write the Dyson series in more convinient form

$$U(t) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t \cdots \int dt_1 \ldots dt_n \; T\left(\hat{V}(t_1) \ldots \hat{V}(t_n)\right) \equiv T \exp\left[-i\int_0^t dt_1 \, \hat{V}(t_1)\right]$$

The exponential form is defined as the series, but it is indeed a nice and compact way to state the answer.

(5) S-matrix : - short for scattering matrix

- connects in-going to out-going states
- that is ⇒ S-matrix time evolves the wave function

$$S(t_1, t_2)\, \hat{\Psi}(t_2) = \hat{\Psi}(t_1)$$

- Basically if I have the S-matrix, problem is solved, almost ...
- using the $U(t)$ operator, we can express the S-matrix as follows

$$\hat{\Psi}(t_1) = U(t_1)\,\hat{\Psi}(0) = S(t_1, t_2)\,\hat{\Psi}(t_2) = S(t_1, t_2)\, U(t_2)\,\hat{\Psi}(0)$$

$$= \underbrace{U(t_1)\, U^{+}(t_2)}_{S(t_1,t_2)}\, U(t_2)\,\hat{\Psi}(0)$$

$U^{+} = U^{-1}$ since U is unitary

Properties of the S-matrix:

(1) $S(t,t) = U(t)\,U^{+}(t) \equiv 1$ [No need to evolve the wave function is time does not change]

(2) $S^{+}(t_1, t_2) = \left[U(t_1)\,U^{+}(t_2)\right]^{+} = U(t_2)\,U^{+}(t_1) = S(t_2, t_1)$

(3) $S(t_1, t_2)\, S(t_2, t_3) = U(t_1)\, \cancel{U^{+}(t_2)}\, \cancel{U(t_2)}\, U^{+}(t_3) = S(t_1, t_3)$

(4) $\partial_{t_1} S(t_1,t_2) = \partial_{t_1} U(t_1)U^+(t_2) = -i\hat{V}(t_1)U(t_1)U^+(t_2) = -i\hat{V}(t_1)S(t_1,t_2)$

Hence $S(t_1,t_2) = T \exp\left[-i\int_{t_2}^{t_1} dt'\, \hat{V}(t')\right]$

(6) Gell-Man Low formalism :

- So far we have a formalism to evolve wave functions in time
- We still need the initial wave functions !

Solution :

- Let ϕ_0 be the ground state of H_0
- Let us set $\hat{\Psi}(-\infty) = \phi_0$ [that is in the "dim past"]
- Turn on the interactions slowly

$\Rightarrow$ $\hat{\Psi}(t)$ evolves adiabatically

$\Rightarrow$ $\hat{\Psi}(t)$ becomes the ground state of $H = H_0 + V$ in the present

$\Rightarrow$ $\hat{\Psi}(t) \equiv S(t,-\infty)\hat{\Psi}(-\infty)$

- To compute operators $\langle \hat{O}(t) \rangle$, we also need the "bra"

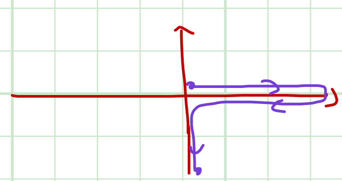
$\Rightarrow$ "cheat" turn off the interactions slowly so that in the dim future $\hat{\Psi}(\infty) \to \phi_0$

$\Rightarrow \langle \hat{\Psi}^+(\infty) S(\infty,t)\, \hat{O}(t)\, S(t,-\infty)\hat{\Psi}(-\infty)\rangle$ [Gell-Mann Low]

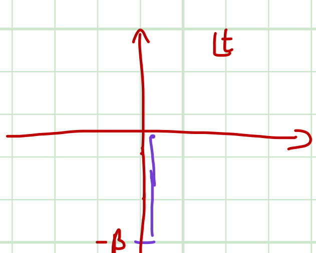
[Note : there are alternative formalisms where "cheating" is not required. We will return to this topic once we are comfortable with diagrams]

Gell-Man Low contour

Keldysh



Matsubara



$\Rightarrow$ thermodynamics by working in Im time

No need to go to infinite future, return to past

$\Rightarrow$ combine real time evolution with thermodynamics